A UNIQUENESS THEOREM FOR HAAR AND WALSH SERIES(1)

BY
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1. **Introduction.** It is well known that if a trigonometric series converges to an integrable function, except possibly in a countable set, and the series' coefficients converge to zero then that series is the Fourier series of the limit function [8, p. 329].

This problem for Walsh series was open for many years, but in 1965 two independent solutions were published: [2] and [3]. By combining the American and Soviet techniques we are able to obtain a theorem which contains the Walsh series result in [2] but which has a form similar to the theorem in [3]. Following the Soviet example, we will prove the result for Haar series and obtain the Walsh series result as a corollary.

In this paper E will represent a countable subset of [0, 1], and D.R. will represent the set of dyadic rationals. Given a Borel set A, $I_A(x)$ will denote the characteristic function of the set A.

The Haar system $\{\chi_k\}_{k=0}^{\infty}$ is defined as $\chi_0(x)=1$, $\chi_1(x)=I_{[0,1/2)}(x)-I_{(1/2,1]}(x)$; in general writing $k'=2^n+k$, $0\leqq k<2^n$ where n is the largest power of 2 which is less than or equal to k', we define

$$\begin{aligned}\chi_k(x) = \chi_n^{(k)}(x) &= (2^n)^{1/2} && 2k-2/2^{n+1} < x < 2k-1/2^{n+1},\\ &= -(2^n)^{1/2} && 2k-1/2^{n+1} < x < 2k/2^{n+1},\\ &= (2^n/4)^{1/2} && x = k-1/2^n,\\ &= -(2^n/4)^{1/2} && x = k/2^n,\\ &= 0 && \text{otherwise.}\end{aligned}$$

We will denote

$$(2k-2/2^{n+1}, 2k-1/2^{n+1}) = \Delta_{k'}^{(1)}, \qquad (2k-1/2^{n+1}, 2k/2^{n+1}) = \Delta_{k'}^{(2)}$$

and these two open intervals will be referred to as the positive (respectively negative) support of the kth Haar function. Alexits [1] proves this sequence is a complete orthonormal system.

We define the Walsh system $\{\Psi_k\}_{k=0}^{\infty}$ by letting $\Phi_0(x)=I_{[0,1/2)}(x)-I_{[1/2,1]}(x)$, $\Phi_n(x)=\Phi_0(2^n x)$ (where Φ_0 is extended by periodicity of period 1) and then defining $\Psi_0(x)=1$, $\Psi_n(x)=\Phi_{n_1}(x)\cdots\Phi_{n_r}(x)$ where $n=\sum_{i=1}^{r} 2^{n_i}$ and the n_i are uniquely

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determined by $n_{i+1} < n_i$. Walsh [7] proves this sequence is a complete orthonormal system.

Both the Walsh and Haar systems are extended by periodicity of period 1 to the whole real line. Kacmarzc noted [1, p. 62] that if $k' = 2^n + k$, $0 \leqq k < 2^n$ and n largest such that $2^n \leqq k'$, then

$$\Psi_{k'}(x) = \frac{1}{(2^n)^{1/2}} \sum_{j=1}^{2^n} \varepsilon_{k'j} \chi_n^{(j)}(x) \quad \text{for } x \notin \text{D.R.} \tag{1}$$

where $\varepsilon_{jk'} = \pm 1$ and the matrix $[\varepsilon_{jk'}]$ has orthogonal rows. If S is a Haar or Walsh series, S_n will denote the partial sum of order $n-1$.

Given a Haar series $S(x) = \sum_{k=0}^{\infty} a_k \chi_k(x)$, we shall say S satisfies *condition* G if for every $x_0 \in [0, 1]$,

$$\lim_{j \to \infty} \frac{a_{k_j}}{\chi_{k_j}(x_0)} = 0 \tag{2}$$

where k_j are all those integers p for which $\chi_p(x_0) \neq 0$. We note that this growth condition is essential to Theorem 1 since

$$\chi_0(x) + \sum_{n=0}^{\infty} (2^n)^{1/2} \chi_{2^n}(x) \equiv 0 \quad \text{on } (0, 1]$$

but the series fails to satisfy (2) only at the point $x_0 = 0$.

We shall say a function g is in the class $\mathscr{A}$ if there is a closed countable set $A_g \subseteq [0, 1]$ such that g is locally integrable in $(0, 1) \sim A_g$. We note that $L^1[0, 1]$ is a proper subset of $\mathscr{A}$ since $A_{\{1/x\}} = \{0\}$.

We shall prove:

THEOREM 1. *Let* $S(x) = \sum_{k=0}^{\infty} a_k \chi_k(x)$ *be a Haar series satisfying condition* G *such that for some function* $g \in \mathscr{A}$ *and subsequence of natural numbers* $\{n_j\}$

(i) $\lim_{j \to \infty} S_{2^{n_j}}(x) = g(x)$ *in measure*;

(ii) $\limsup_{j \to \infty} |S_{2^{n_j}}(x)| < \infty$, $x \notin E$;

(iii) $\limsup_{j \to \infty} S_{2^{n_j}}(x)$ *dominates an integrable function* $f(x)$ *for* $x \notin E$.

Then g *is integrable and* S *is the Haar Fourier series of* g.

THEOREM 2. *Let* $S(x) = \sum_{k=0}^{\infty} a_k \Psi_k(x)$ *be a Walsh series such that for a function* $g \in \mathscr{A}$ *and some subsequence of natural numbers* $\{n_j\}$

(i) $\lim_{j \to \infty} S_{2^{n_j}}(x) = g(x)$ *in measure*;

(ii) $\limsup_{j \to \infty} |S_{2^{n_j}}(x)| < \infty$ *for* $x \notin E$;

(iii) $\limsup_{j \to \infty} S_{2^{n_j}}(x)$ *dominates an integrable function* $f(x)$ *for* $x \notin E$;

(iv) $\lim_{k \to \infty} a_k = 0$.

Then g *is integrable and* S *is the Walsh Fourier series of* g.

2. **Fundamental lemmas.** By $\alpha_n = \alpha_n(x)$ and $\beta_n(x) = \beta_n$ we shall mean

$$\alpha_n = P \cdot 2^{-n} \leqq x < (P+1)2^{-n} = \beta_n \tag{3}$$

and $\alpha_n'(x)=\alpha_n(x)$ if $x \notin$ D.R., $\alpha_n'(x)=\alpha_n(x)-2^{-n}$ otherwise. Given a Haar series $S(x)=\sum_{k=0}^{\infty} a_k\chi_k(x)$ we define

$$L(S, x, n_j) = \lim_{j\to\infty} \sum_{k=0}^{2^{n_j}-1} a_k \int_0^x \chi_k(u)\, du = \lim_{j\to\infty} L_{2^{n_j}}(S, x) \tag{4}$$

when this limit exists. In case $L(S, x, n)$ exists we will write it as $L(S, x)$ following the notation in [5].

LEMMA 1. *If $S(x)=\sum_{k=0}^{\infty} a_k\chi_k(x)$ is a Haar series satisfying condition G, then for every x and $n\geqq 0$ both $L(S, \alpha_n(x))$ and $L(S, \beta_n(x))$ exist and are finite. Furthermore*

$$L(S, \beta_n(x))-L(S, \alpha_n(x)) = 2^{-n}S_{2^n}(x), \tag{5}$$

$$L(S, \beta_n(x))-L(S, \alpha_n(x)) = o(1) \quad \textit{as } n\to\infty. \tag{6}$$

Proof. We use [3, Lemma 1] as an outline of the proof of this lemma. To show (6), we use (5) and the fact that *condition G* implies $2^{-n}S_{2^n}(x)=o(1)$.

To see this, given $\varepsilon>0$ and $x_0\in[0, 1]$ we choose N sufficiently large so that $j\geqq N$ implies

$$|a_{k_j}| < \varepsilon\cdot|\chi_{k_j}(x_0)| \tag{7}$$

where k_j are defined in the definition of *condition G*. We then recall from the very definition of the Haar functions that the set

$$\{\chi_k^{(1)}(x_0), \chi_k^{(2)}(x_0), \ldots, \chi_k^{(2^k)}(x_0)\} \tag{8}$$

has at most two nonzero elements for each k, and that $|\chi_k^{(p)}(x_0)|^2\leqq 2^k$.

Combining (7) and (8) if $n>N$,

$$\left|\frac{S_{2^n}(x_0)}{2^n}\right| \leqq \sup_{0\leqq j\leqq N} \frac{|a_{k_j}|\cdot N}{(2^n)^{1/2}}+2\varepsilon\cdot\frac{\sum_{k=0}^{n} 2^k}{2^n}. \tag{9}$$

Hence taking lim sup of (9) as $n\to\infty$ we obtain $S_{2^n}(x_0)=o(2^n)$.

The following lemma is the Haar series analogue of a lemma in [3], and the proof is essentially the same.

LEMMA 2. *If $S(x)=\sum_{k=0}^{\infty} a_k\chi_k(x)$ is a Haar series satisfying condition G and if $L(S, x, n_j)$ exists and is finite, then*

$$L(S, \alpha_{n_j}') \to L(S, x, n_j) \quad \textit{as } j\to\infty, \tag{10}$$

$$L(S, \beta_{n_j}) \to L(S, x, n_j) \quad \textit{as } j\to\infty. \tag{11}$$

The following lemma is proved in a similar manner to Lemma 3 of [3].

LEMMA 3. *Let $G(x)$ be defined in $a<x<b$ and satisfy*:

(i) *Except perhaps on a countable set Z*, $\liminf_{j\to\infty} 2^{n_j}[G(\beta_{n_j})-G(\alpha_{n_j})]\leqq 0$.

(ii) *For all $x\in(a, b)$, both $G(\alpha_{n_j}')\to G(x)$, $G(\beta_{n_j})\to G(x)$ as $j\to\infty$.*

Then $G(x)$ is monotone nonincreasing in (a, b).

LEMMA 4. *Let $S(x)=\sum_{k=0}^{\infty} a_k\chi_k(x)$ be a Haar series satisfying condition G, let f be a finite valued function, integrable over $(a-\varepsilon, b+\varepsilon)$ for $\varepsilon>0$, and let $\{n_j\}$ be a subsequence of the natural numbers. Suppose further that Z is a countable subset of $(a-\varepsilon, b+\varepsilon)$ and that*

(i) *$L(S, x, n_j)$ exists and is finite for every $x\in(a-\varepsilon, b+\varepsilon)$;*

(ii) *$+\infty>\limsup_{j\to\infty} S_{2^{n_j}}(x)\geqq f(x)$ for $x\notin Z$.*

Then

$$\limsup_{j\to\infty} S_{2^{n_j}}(x) = \liminf_{j\to\infty} S_{2^{n_j}}(x) \quad \text{a.e. in } (a, b) \tag{12}$$

and this function is integrable over (a, b).

Proof. From hypothesis (ii) and (5) we have

$$\limsup_{j\to\infty} 2^{n_j}[L(S, \beta_{n_j})-L(S, \alpha_{n_j})] \geqq f(x) \quad \text{for } x\in(a-\varepsilon, b+\varepsilon) \sim Z. \tag{13}$$

Using the Vitali-Carathéodory Theorem [5, p. 75] we choose an upper semi-continuous function $f_1\leqq f$ in $L^1(a-\varepsilon, b+\varepsilon)$.

Let $F_1(x)=\int_{a-\varepsilon/2}^{x} f_1(u)\,du$ for $x\in(a-\varepsilon/2, b+\varepsilon)$. Clearly since $|\alpha_{n_j}-\beta_{n_j}|=2^{-n_j}$,

$$\liminf_{j\to\infty} 2^{n_j}[F_1(\beta_{n_j})-F_1(\alpha_{n_j})] \leqq f(x) \quad \text{for } x\in(a-\varepsilon/2, b+\varepsilon).$$

Thus by (13), for $x\in(a-\varepsilon/2, b+\varepsilon)\sim Z$,

$$\liminf_{j\to\infty} 2^{n_j}\{F_1(\beta_{n_j})-L(S, \beta_{n_j})-[F_1(\alpha_{n_j})-L(S, \alpha_{n_j})]\} \leqq 0. \tag{14}$$

Now Lemma 2 and continuity of F_1 imply hypothesis (ii) of Lemma 3 is also satisfied by the function $F_1(x)-L(S, x, n_j)$, so by Lemma 3 this function is monotone nonincreasing in $(a-\varepsilon/2, b+\varepsilon)$.

Thus $L(S, x, n_j)$ has derivatives almost everywhere which are integrable on every closed subinterval of $(a-\varepsilon/2, b+\varepsilon)$. Hence (12) is a consequence of the equality

$$\frac{L(S, \beta_{n_j})-L(S, \alpha_{n_j})}{\beta_{n_j}-\alpha_{n_j}} = S_{2^{n_j}}(x). \tag{15}$$

LEMMA 5. *If $S'(x)=\sum_{k=0}^{\infty} c_k\chi_k(x)$ is the Haar Fourier series of an integrable function $f(x)$ then*

$$S' \text{ satisfies condition } G, \tag{16}$$

$$\lim_{n\to\infty}\int_0^1 |S_n'(x)-f(x)|\,dx = 0. \tag{17}$$

Proof. By assumption $c_k=\int_0^1 f(x)\chi_k(x)\,dx$. Thus

$$|c_k| \leqq \max_x |\chi_k(x)|\left\{\int_{\Delta_k^{(1)}\cup\Delta_k^{(2)}} |f(x)|\,dx\right\}.$$

But f is integrable, so $m(\Delta_k^{(1)}\cup\Delta_k^{(2)})\to 0$ implies

$$\frac{|c_k|}{\max_x |\chi_k(x)|}\to 0 \quad \text{as } k\to\infty.$$

But if $\chi_k(x_0)\neq 0$ then $|\chi_k(x_0)|\geqq\frac{1}{2}\max_x|\chi_k(x)|$ so (16) is proved. (17) is the theorem appearing in [6].

We quote the main lemma of [2].

LEMMA 6. *Suppose* $S(x)=\sum_{k=0}^{\infty}a_k\chi_k(x)$ *is a Haar series satisfying condition G and f is an integrable function whose Haar Fourier series is* $S'(x)=\sum_{k=0}^{\infty}c_k\chi_k(x)$. *Suppose also that for some subsequence of natural numbers* $\{n_j\}$, $S_{2^{n_j}}(x)$ *converges to* $f(x)$ *in measure.*

Let $x_0\in[0,1]$ and k_0 be an integer such that

(α) inside $\Delta_{k_0}^{(i_0)}$, $i_0=1$ or 2, $S_{k_0+1}(x)$, $S'_{k_0+1}(x)$ have different constant values,

(β) $k>k_0$ implies χ_k is nonzero either in $\Delta_{k_0}^{(i_0)}$ or outside it.

Then for any $M>0$ and any positive integer N we can find an n_j, a natural number p and an interval of the form $\Delta_p^{(i_p)}$, $i_p=1$ or 2 such that

(1) $2^{n_j}>N$,

(2) $x_0\notin\Delta_p^{(i_p)-}$ and $\Delta_p^{(i_p)-}\subset\Delta_{k_0}^{(i_0)}$,

(3) $|S_{2^{n_j}}(x)|>M$ for $x\in\Delta_p^{(i_p)}$ and is constant there,

(4) p, $\Delta_p^{(i_p)}$ satisfy (α), (β).

We close this section by recalling a theorem of Vitali [4, p. 152].

LEMMA 7. *If* $\{u_n\}_{n=0}^{\infty}$ *is a sequence of functions which have equiabsolutely continuous integrals; i.e., given* $\varepsilon>0$ *there is a* δ *such that for* $E\subseteq[0,1]$, $m(E)<\delta$ *implies*

$$\left|\int_E u_n(x)\,dx\right| < \varepsilon \quad \textit{for all } n;$$

and if $u_n(x)\to f(x)$ *in measure, then* $f(x)$ *is integrable and*

$$\lim_{n\to\infty}\int_0^1 u_n(x)\,dx = \int_0^1 f(x)\,dx.$$

3. **Proof of Theorem 1.** Since $g\in\mathscr{A}$, given $x_0\notin A_g$ we can find an interval of the form $\Delta_{j_0}^{(i)}$ containing x_0 such that $\Delta_{j_0}^{(i)}\cap A_g=\varnothing$, and g is integrable over $\Delta_{j_0}^{(i)}$. We claim that $L(S,x_0,n_j)$ exists and is finite. By Lemma 1, we may assume x_0 is not a dyadic rational.

Consider the function

$$g^*(x) = [g(x)-S_{j_0+1}(x)]I_{\Delta_{j_0}^{(i)}}(x). \tag{18}$$

By choice of j_0, g^* is integrable in [0, 1]. By hypothesis (i), $S_{2^{n_j}}(x)I_{\Delta_{j_0}^{(i)}}(x)\to g(x)\cdot I_{\Delta_{j_0}^{(i)}}(x)$ in measure, so if we define

$$T(x) = [S(x)-S_{j_0+1}(x)]I_{\Delta_{j_0}^{(i)}}(x) \tag{19}$$

we have by (18)

$$\lim_{j\to\infty} T_{2^{n_j}}(x) = g^*(x) \quad \text{in measure.} \tag{20}$$

We note that T is actually a Haar series, say $T(x)=\sum_{k=0}^{\infty}\alpha_k\chi_k(x)$, and since $|\alpha_k|\leqq|a_k|$, T satisfies *condition G*. Let $S'(x)=\sum_{k=0}^{\infty}c_k\chi_k(x)$ be the Haar Fourier series of g^*. We claim S' and T are the same series.

Suppose not, and let k_0 be the least integer for which $\alpha_{k_0}\neq c_{k_0}$. Clearly $T_{k_0+1}(x) - S'_{k_0+1}(x)\equiv d$ in $\Delta_{k_0}^{(i_0)}$ where d is a nonzero constant and $i_0=1$ or 2.

Let $E\cup$ D.R. $=\{Z_1, Z_2, \ldots\}$; we use Lemma 6 countably many times to obtain sequences $\{n_{j_k}\}_{k=1}^{\infty}$, $\{p_k\}_{k=1}^{\infty}$ such that

$$Z_k\notin\Delta_{p_k}^{(i_{p_k})^-}\subset\Delta_{p_{k-1}}^{(i_{p_{k-1}})}\quad\text{for } k=2,3,\ldots, \tag{21}$$

$$|T_{2^{n_{j_k}}}(x)|>k\quad\text{for } x\in\Delta_{p_k}^{(i_{p_k})},\qquad k=1,2,\ldots. \tag{22}$$

Since the dyadic rationals are excluded from the nested sequence $\Delta_{p_k}^{(i_{p_k})}$, $m(\Delta_{p_k}^{(i_{p_k})})$ tends to zero as $k\to\infty$. By (21) we let $\xi\in\bigcap_{k=1}^{\infty}\Delta_{p_k}^{(i_{p_k})}$. Then by (22),

$$\limsup_{k\to\infty}|S_{2^{n_{j_k}}}(\xi)|=\infty$$

which implies by hypothesis (ii) that $\xi\in E\subseteq\{Z, Z_2,\ldots\}$ which contradicts (21). Thus the assumption was false and $S'\equiv T$.

Thus by (19), if $n>j_0+1$ and $x\in\Delta_{j_0}^{(i)}$,

$$S_n(x)=T_n(x)+S_{j_0+1}(x). \tag{23}$$

Since T is a Fourier series and S_{j_0+1} is a polynomial we use (17) and hypothesis (i) to see S_n satisfies the hypotheses of Lemma 7 inside $\Delta_{j_0}^{(i)}$. Choose $\rho\in$ D.R. $\cap\,\Delta_{j_0}^{(i)}$ to the left of x_0, and use Lemma 7 to obtain

$$\begin{aligned}L(S,x_0,n_j)-L(S,\rho)&=\lim_{j\to\infty}\int_{\rho}^{x_0}S_{2^{n_j}}(u)\,du\\&=\int_{\rho}^{x_0}g(u)\,du<\infty\end{aligned}$$

since $(\rho,x_0)\subseteq\Delta_{j_0}^{(i)}$. But by Lemma 1, $L(S,\rho)$ is finite and thus $L(S,x_0,n_j)$ exists and is finite. The claim is thus true.

Let $N=\{x\in(0,1)\mid L(S,x,n_j)$ does not exist$\}$. By our claim $N\subseteq A_g$. If we can show that N has no isolated points, then $\bar{N}\subseteq(A_g)^-=A_g$ would be a perfect countable set, which would force N to be empty. But by (4) and orthogonality of the Haar functions $L(S,0)=0$, $L(S,1)=a_0$, and so we will have shown that $L(S,x,n_j)$ exists and is finite everywhere in $[0,1]$.

Suppose indeed that N has an isolated point x_0. Then for numbers $0<a<c<d<b<1$, $L(S,x,n_j)$ exists and is finite in $[c,x_0)\cup(x_0,d]$. Choose f_1 by the Vitali-Carathéodory Theorem as in Lemma 4, and define $F_1(x)=\int_0^x f_1(u)\,du$. Using (15), Lemma 2, hypothesis (ii), and continuity of F_1,

$$\liminf_{j\to\infty}2^{n_j}\{F_1(\beta_{n_j})-L(S,\beta_{n_j})-[F(\alpha_{n_j})-L(S,\alpha_{n_j})]\}\leqq 0\quad\text{for } x\notin E, \tag{24}$$

$$\begin{aligned}F_1(\alpha'_{n_j})-L(S,\alpha'_{n_j})&\to F_1(x)-L(S,x,n_j),\\F_1(\beta_{n_j})-L(S,\beta_{n_j})&\to F_1(x)-L(S,x,n_j).\end{aligned} \tag{25}$$

Thus by Lemma 3, $F_1(x)-L(S, x, n_j)$ is monotone nonincreasing in each of the intervals (c, x_0), (x_0, d). But F_1 is continuous so $L(S, x, n_j)$ has a right and left limit (in the extended real plane) as $x \to x_0$. By (6) these limits must be equal, and by monotonicity they are finite. Thus by using (4), and the fact that the first 2^{n_j} Haar functions are constant in $(\alpha_{n_j}, \beta_{n_j})$

$$\begin{aligned} |L_{2^{n_j}}(S, x_0)-L_{2^{n_j}}(S, \alpha_{n_j})| &= \left| \int_{\alpha_{n_j}}^{x_0} S_{2^{n_j}}(u)\, du \right| \\ &= |(x_0-\alpha_{n_j})\cdot S_{2^{n_j}}(x_0)| \\ &\leqq \frac{|S_{2^{n_j}}(x_0)|}{2^{n_j}} = o(1) \end{aligned}$$

by (9). Thus $L_{2^{n_j}}(S, x_0)=L(S, \alpha_{n_j})+o(1)$ which implies $L(S, x_0, n_j)$ exists and is finite, and we have proved N has no isolated points.

By periodicity of the Haar functions we conclude that $L(S, x, n_j)$ exists and is finite for all real x. Hence using Lemma 4 we conclude $g \in L^1[-1/2, 3/2]$ and $S_{2^{n_j}}(x) \to g(x)$ almost everywhere.

We now proceed as we did locally for g^*. If the Haar Fourier series of the integrable function $g(x)$ is not identically equal to the series S, then using Lemma 6 countably many times we conclude the E is uncountable contrary to hypothesis. Thus S is the Haar Fourier series of the function g and Theorem 1 is established.

4. **Proof of Theorem 2.** Using (1) and defining

$$\alpha_k = \alpha_n^{(k')} = \sum_{i=2^n}^{2^{n+1}-1} \frac{\varepsilon_{ik}}{(2^n)^{1/2}} a_i, \tag{26}$$

we see that the Haar series

$$T(x) = \sum_{k=0}^{\infty} \alpha_k \chi_k(x)$$

satisfies

$$T_{2^n}(x) = S_{2^n}(x) \quad \text{for } x \notin \text{D.R.}$$

Thus the Haar series T satisfies hypotheses (i), (ii) and (iii) of Theorem 1, for the countable set $E \cup$ D.R. We now claim that hypothesis (iv) implies T satisfies *condition G*. To see this use (26) to obtain

$$|\alpha_n^{(k)}| \leqq (2^n)^{1/2} \max_{2^n \leqq i < 2^{n+1}} |a_i|. \tag{27}$$

But by definition of the Haar functions, $\max_x |\chi_n^{(k)}(x)| = (2^n)^{1/2}$, so we use the argument in Lemma 5 to conclude T satisfies *condition G*.

Thus by Theorem 1, g is integrable, and T is its Haar Fourier series. We now observe that (26) implies

$$a_k = \sum_{i=2^n}^{2^{n+1}-1} \frac{\varepsilon_{ik}}{(2^n)^{1/2}} \alpha_i. \tag{28}$$

Hence by (1), (28) and Theorem 1,

$$\begin{aligned} a_k &= \sum_{i=2^n}^{2^{n+1}-1} \frac{\varepsilon_{ik}}{(2^n)^{1/2}} \int_0^1 g(x)\chi_i(x)\,dx \\ &= \int_0^1 g(x) \sum_{i=2^n}^{2^{n+1}-1} \frac{\varepsilon_{ik}\chi_i(x)}{(2^n)^{1/2}}\,dx \\ &= \int_0^1 g(x)\Psi_k(x)\,dx \end{aligned}$$

which means that S is the Walsh Fourier series of the integrable function g.

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